Note on Immune Network and Majority Network *

Yoshiteru Ishida †

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Abstract

State Propagation Net is proposed by motivations for simplifying an immune network model. *State Propagation Net* is a certain class of graph whose nodes have two states: active and inactive. The behavior of *State Propagation Net* is discussed by graph theoretical characterization. Although *State Propagation Net* is comparable with *Life Game*, *Cellular Automata*, and *Majority Net*, we discuss its relation with *Majority Net* in this paper. It is shown that *Majority Net* turns out to be a proper subset of *State Propagation Net*.

1 Introduction

Immune networks [1, 2] and neural networks [3] as a computational model for pattern recognition and learning can be viewed as a large-scale non-linear network. These systems are, in principle, the network consisting of the homogeneous unit (or small set of different types), and one (or small set of types) of the connection. Unit (or different types of units) and connection can be regarded as a finite automaton, and hence characterized by the state-transition function. The next state of unit and connection will be determined by the states of adjacent nodes and connections.

Most of the immune network models inspired by the concept of idiotype network [4] are continuous differential equations describing the population dynamics [5, 6] of B-cell, antibody and T-cell. However, using continuous differential equations is not always adequate;

- In order to attain the intelligent nature by parallel distributed processing by the huge number of units, the continuous differential equations are too difficult to analyze.
- In order to study information processing capabilities such as memory, pattern recognition and learning, discrete model may be more appropriate to focus on the qualitative behavior of the systems.

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†Graduate School of Information Science, Nara Institute of Science and Technology, 8916-5 Takayama, Ikoma, Nara, 630-01 Japan; email: ishida@is.aist-nara.ac.jp

Several computational models of immune systems have been recently proposed [7, 8] for the study of information processing models based on immune systems.

In this note, we concentrate on the analysis of behavior of the state propagation in graph [9, 10] that is obtained by several simplifications on Richter model[12]. For the study of machine learning, the parameter updating scheme for the state-transition functions of node as well as arcs must be considered. However, learning capability of the model is out of scope of this note. We also discuss the relation between our model and majority net. Agur [11] have also discussed the majority rule cellular automaton relating with immune systems.

For the study of immune networks, additional dynamic nature may be necessary: that is the generation of new units and/or deletion of the existing units according to some rules. If this dynamic nature should be taken into accounts, the existing approach with continuous dynamical systems seems almost impossible for large number of units. Although this note does not deal with the model of the variable number of units, we aim to analyze such discrete dynamical models in the future.

In section 2, some preliminaries including the definition of *State Propagation Net* is presented. Section 3 discusses the behavior of the *State Propagation Net* using the characteristic set defined on the signed directed graph. Section 3 also presents the method for analyzing the behavior of *State Propagation Net*. Section 4 discusses the relation between *State Propagation Net* and *Majority Net*.

2 Preliminaries

2.1 The state propagation net

Definition 2.1 *Signed Directed Graph*
The signed directed graph is a triplet (V, PA, NA) where V is a set of vertices, PA is a set of positive arcs $PA = V \times V$, and NA is a set of positive arcs $NA = V \times V$.

The following notations are used in this note.
$\Gamma_+(x_i) = \ \{x_j : (x_i, x_j) \in PA\}$
$\Gamma_-(x_i) = \{x_j : (x_i, x_j) \in NA\}$
$\Gamma_+^{-1}(x_i) = \{x_j : (x_j, x_i) \in PA\}$
$\Gamma_+^k(x_i) = \{x_{ik} : (x_i, x_{i1}), (x_{i1}, x_{i2}), .., (x_{ik-1}, x_{i1k}) \in PA\}$
$\Gamma_+(S_i) = \{x_j : (x_i, x_j) \in PA, x_i \in S_i\}$

Definition 2.2 *State Propagation Net*
State Propagation Net (VS, AS, VF, VA) *is defined on the signed directed graph* (V, PA, NA). *VS and AS are set of finite states for the set of vertices and the set of arcs respectively. VF and VA are state transition functions for the set of vertices and the set arcs.*

In this note, we restrict ourselves that the state of arcs has only two state: positive and negative, that are given by the signed directed graph, and does not change at all. The state of the vertices will change based on the current state of the adjacent arcs and vertices.

Definition 2.3 *Active Set*
Let $A(S_i)$ denote the potentially active set when current active set is S_i. In this note, we define $A(S_i) = \{x_k : |\Gamma_+^{-1}(x_k) \cap S_i| > |\Gamma_-^{-1}(x_k) \cap S_i|\}$. In the same manner, the potentially inactive set $IA(S_i)$ can be defined; $IA(S_i) = \{x_k : |\Gamma_+^{-1}(x_k) \cap S_i| < |\Gamma_-^{-1}(x_k) \cap S_i|\}$.

That is, the potentially active set consists of such nodes that the number of incoming positive arcs from S_i is greater than that of incoming negative arcs from S_i. The potentially active set $A(S_i)$ can be comparable with the reachable set by one step of arcs: $\Gamma(S_i)$ in the usual graph with single-type arc. By identifying the activated sets with the reachable sets in this manner, the propagation in the usual graph with single-type arc has linearity. The significant difference in propagation through *State Propagation Net* from that of usual graph with single-type arc is two-fold:

- Non Monotonicity: $A^k(S) \subseteq A^{k+1}(S)$ does not hold in general.
- Non Additivity: $A(S_i \cup S_j) = A(S_i) \cup A(S_j)$ does not hold in general.

There are two ways of activating potentially active nodes: asynchronous and synchronous. Asynchronous version activates the potentially active nodes and inactivates the potentially inactive nodes one by one, evaluating active and inactive one after each activation/inactivation. In contrast, synchronous version activates all the potentially active nodes and inactivates all the potentially inactive nodes simultaneously. In this note, we focus on the synchronous version of the model. Asynchronous version of the model can be translated into Petri-Net. Synchronous version can be translated into the propagation in the graph defined on the power set of the node of the original graph, and the arc from S_i to S_j exists only when $S_j = S_i \cup A(S_i) - S_i \cap IA(S_i)$.

Although *State Propagation Net* can be theoretically identical with the graph defined on the power set as mentioned above, this approach is impractical for the large-scale net we are concerned, since the number of nodes become 2^N where N is the number of nodes of original state propagation net. The synchronous version of *State Propagation Net* model can also be implemented by the comparator network where the comparator node can compare the number of positive incoming arcs and negative incoming arcs and activate itself depending upon the comparison.

2.2 Characteristic sets

Definition 2.4 *(positive/negative dominance)*
Positive (negative) indegree of a vertex is the indegree by the positive (negative) arcs. A

vertex is called **positive(negative) dominant** *when the positive(negative) indegree is equal to or greater than the negative (positive) indegree. A vertex is called* **strictly positive(negative) dominant** *when the positive(negative) indegree is greater than the negative (positive) indegree.*

Definition 2.5 *(internal dominance)*
A set of vertices $S \subseteq V$ is called **positive(negative) internal dominant** [1] *when all the vertices in the subgraph induced by the set S are positive(negative) dominant.*

Definition 2.6 *(external dominance)*
A set of vertices $S \subseteq V$ is called **positive(negative) external dominant** *when all the vertices in V -S in the subgraph obtained by deleting all the arcs in S and V-S (i.e., only arcs between S and V-S are remained) are positive(negative) dominant.*

3 Analysis and Classification of the Behavior

3.1 Classification by the pattern of activated sets

The behavior of the systems described by *State Propagation Net* discussed above falls upon in the following three types.

- Periodic[2]: After k steps of activation, the activated set will be identical with the initial activated set. We call the smallest integer k the period.
- Stable: The activated set does not change. Although this behavior can be regarded as **periodic** with period 0, we separate this special periodic behavior from the above periodic classes. Clearly, the null set is stable for all the systems described by *State Propagation Net*.
- Transient: The initial activated set will eventually become the above two types of attractors: periodic or stable. In case of the behavior transient to the stable set, we will further notice the following two types:

[1] The concept of **positive/negative internal dominance** can be related to the concept of **internal stability**[13] in the non-oriented graph. A set of vertices S is called internal stable when any vertex in S is not adjacent to the other vertices in S. Likewise, a set of vertices S is positive internal dominant when any vertex in S is not strictly negative dominant from S. The concept of **positive(negative) external dominance** can also be related to the concept of **external stability**. The set of vertices S is external stable when it can cover whole set of vertices. Likewise, a set of vertices S is positive external dominant when it can cover the whole set of vertices by positive dominance. We do not use the word "stable" for these concepts to avoid the confusion with the "stability" concept that will be defined in the next section.

[2] The terminology used to describe the behavior in this and the next subsection is mostly from the analogy from the continuous dynamical systems for conveniences of the researchers in the continuous dynamical systems.

– Divergence: When the stable set is all the set of vertices in the graph.

– Convergence: When the stable set is some finite set of vertices including the null set, but not the whole set of vertices.

Example 3.1

Figure 1 shows an example of *State Propagation Net*. When $\{x_3, x_4\}$ is the initial activated set, the behavior will be **periodic** with period one. That is, the subset $\{x_3, x_4\}$ and $\{x_5, x_6\}$ will be activated alternately. When $\{x_3, x_4, x_5, x_6\}$ is the initial activated set, it does not change at all, hence the behavior is **stable** one. When $\{x_2\}$ is the initial activated set, the behavior will be transient one to the periodic cycle stated above. When $\{x_1\}$ is the initial activated set, the behavior will be transient to the stable attractor stated above.

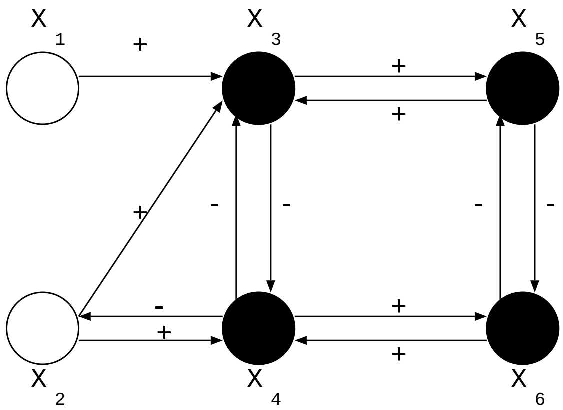

Figure 1: An Example of a State Propagation Net Indicating Stable Pattern. Black node indicates active, and white inactive state.

3.2 Characterization of behavior by graphs

In order to characterize the behavior described by the previous subsection, we first analyze when the activated set becomes stable. The following two theorems follow immediately from the definitions of *State Propagation Net* and positive/negative, internal/external dominances.

Theorem 3.1 *(stable set)*
A set of vertices $S \subseteq V$ is stable if and only if (1) S is positive internal dominant, and (2) S is negative external dominant.

Further, we need to know how the non-stable (i.e., periodic or transient) set will change in the next step.

Theorem 3.2 *(transient set)*
If a set of vertices $S_k \subseteq V$ is the active set, then the active set in the next step S_{k+1} satisfies: (1) S_{k+1} is positive dominant from S_k, and (2) $S_k - S_k \cup S_{k+1}$ is negative dominant from S_k.

Remarks: $A(S_k)$ corresponds to S_{k+1} that is positive dominant from S_k. $IA(S_k)$ corresponds to $S_k - S_k \cup S_{k+1}$ that is negative dominant from S_k.

By characterizing the stable set by this theorem, **divergence** or **convergence** to the non-empty stable activated set can be done. For the analysis of divergence, first the whole set of vertices must be checked if it is stable or not. (Notice that the whole set of vertices does not necessarily stable in general.) If it is stable, then let the whole set be S_k, and find the possible activated set at the previous step S_{k-1} by the above theorem. The same procedure will be continued until the procedure does not get new set. We call the sequence thus obtained **divergence sequence**: $S_0, S_1, ..., S_k(= V)$.
Convergence sequence can be obtained in the same manner by selecting S_k to be the stable set on which the active set will converges. Notice that there are many divergence or convergence sequences for the same final stable set S_k, since there are many previous active sets S_{m-1} for the active set S_m .

For the analysis of convergence on the null set, we can use the **disappearing set** that will be characterized as follows:

Theorem 3.3 *(disappearing set)*
A set of vertices $S \subseteq V$ is disappearing if (1) S is negative internal dominant, and (2) S is negative external dominant.

The **convergence sequence** on the null set can be obtained in the same manner as that of **divergence sequence** by selecting the disappearing set to be S_k. There is no need to check if the null set is stable, since the null set is always stable set.

For finding periodic sequence, the initial activated set S_k is set to be non-stable set. Then, obtain the activated set at the previous time S_{k-1} that may not be unique, and/or the active set at the next time S_{k+1} that is unique. This search process is continued until;

- new activated set is identical with any activated set in the sequence so far obtained, or
- new activated set is stable.

Example 3.2

In the example shown in Figure 1, there are five stable sets:

$\{x_3, x_5\}, \{x_4, x_6\}, \{x_1, x_3, x_5\}, \{x_3, x_4, x_5, x_6\}$, and $\{x_1, x_3, x_4, x_5, x_6\}$. For all these stable sets, the conditions (1) and (2) of theorem 3.1 are satisfied.

For the transient state such as $S_k = \{x_2, x_3, x_4\}$, the next active set will be $S_{k+1} = \{x_3, x_4, x_5, x_6\}$. Since S_{k+1} is positively dominant from S_k and that $S_k - S_k \cup S_{k+1} (= \{x_2\})$ is negative dominant from S_k, thus the conditions (1) and (2) of theorem 3.2 are satisfied.

Since the whole set of this example is not stable, let us consider the convergence sequence on the largest stable set $\{x_1, x_3, x_4, x_5, x_6\}$. Let this set be S_k, then the previous set S_{k-1} may be; $\{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_2, x_3, x_5, x_6\}, \{x_2, x_4, x_5, x_6\}, \{x_1, x_4, x_5, x_6\}, \{x_1, x_5, x_6\}$, or

$\{x_1, x_3, x_4\}$. For each of these six subsets, S_{k-2} will be obtained. One of the possible convergence sequence obtained in this manner is: $\{x_1\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_4, x_5, x_6\}$.

4 State Propagation Net and Majority Net

Majority Net is similar to State Propagation Net so far discussed. In this section, we will show that Majority Net is one specific class of State Propagation Net whose graph has a symmetry. The next subsection briefly presents Majority Net which have been extensively studied.

4.1 Majority Net and its behavior

Definition 4.1 *(Majority Net)*
A graph whose nodes are labeled by one state in the state set $\{s_1, s_2, \cdots, s_n\}$ *is called* Majority Net *if the label is updated simultaneously obeying Majority Rule: The new label of a node is that of the majority of its neighbors. The label of a node is assumed to retain its original label if tie occurs.*

We further assume the state set consists of only two symbols; $\{1(active), 0(inactive)\}$, for simplicity and for comparison with State Propagation Net. Remarkable feature about the behavior of Majority Net is stated in the following theorem quoted from [14].

Theorem 4.1 *([15, 16, 14])*
For every graph, and every initial 01-labeling, the sequence of labelings obtained by the majority rule has eventually period at most two.

The above theorem is known to hold for the state set consisting of more than two symbols [16].

Example 4.1

Figure 2 shows one example of Majority Net and its initial labeling. The labeling switches between that shown in Figure 2 and its complementary state obtained by making all the state opposite (i.e. period two).

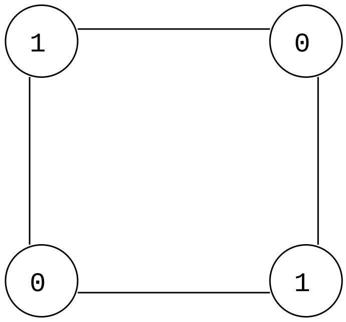

Figure 2: An Example of a Majority Net and Its Initial Labeling

4.2 Relation between State Propagation Net and Majority Net

Although Majority Net having two states: active(1) and inactive(0) looks similar to State Propagation Net, they are not equivalent. In fact, Majority Net is a proper subset of State Propagation Net. This is known from the fact that *State Propagation Net* with period greater than two can be easily constructed and the following transformation scheme from Majority Net to State Propagation Net.

Scheme(Transformation From Majority Net to State Propagation Net)
(1)Network Transformation: Each pair of adjacent nodes in Majority Net shown in the left of Figure 3 is transformed to the four nodes symmetrically interacting shown in the right of Figure 3.
(2)State Transformation: Labeling of Majority Net is translated from 1(0) to the combination of active upper node and inactive lower node(inactive upper node and active lower node). The distinction between lower and upper is ignored after state of all the nodes are translated here.

Example 4.2

Figure 4 shows *State Propagation Net* Transformed from the Majority Net shown in Figure 2. It can be observed that the State Propagation Net has several symmetries originated from the Majority Net; (1) there is no orientation between two adjacent nodes, and (2) label 1 and 0 can be exchanged without changing the qualitative behavior such as the period.

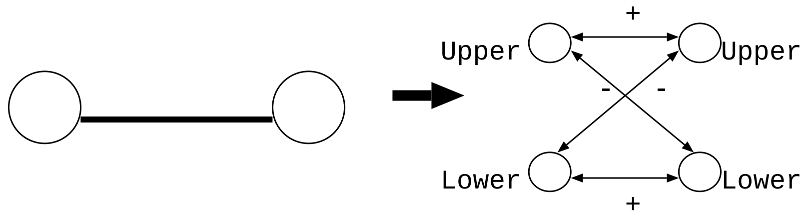

Figure 3: Transformation from Majority Net to State Propagation Net

5 Concluding Remarks

Based on the motivation for simplifying and discretizing the model of population dynamics found in immune network theory, we proposed a graphical model called *State Propagation Net*. The condition for *State Propagation Net* to have stable pattern is characterized graph theoretically. *State Propagation Net* is a finite automaton similar to *Life Game*,*Cellular Automata*, and *Majority Net*, hence the behavior eventually become periodic. *State Propagation Net* is shown to be more general than *Majority Net* that has the mode of period at most two.

Although *State Propagation Net* is motivated by immune network model, it is too unmatured to state any implication on immune systems, and it has been left untouched in this paper. An interesting class of systems from the viewpoint of immune memory in the immune network is that the class where the following memory set can be characterized[3].

> A set of vertices S is memory set when the set S is activated, it will converge on the stable set Q such that $S \cap Q \neq \phi$.

Although *State Propagation Net* is shown to be more complex than *Majority Net* in the sense that it has the mode of longer period, further modification may be needed to analyze the complex networks found in biological systems such as immune networks and ecological systems. In fact, *State Propagation Net* proposed in this paper is just a first step for describing complex networks found in biological systems by discrete systems rather than continuous dynamical systems. In order to make the model more practical, the following modifications should be considered;
(1) Population of the nodes may change dynamically.
(2) Other types of interactions such as inhibition of the dynamic interaction among units

[3] Immune system has a memory in the sense that it will more efficiently eliminate anti-gen in its secondary response than its first encounter. This may be because some units activated in the first encounter are kept active, and they will generate antibodies specific to the anti-gen more efficiently in the secondary encounter.

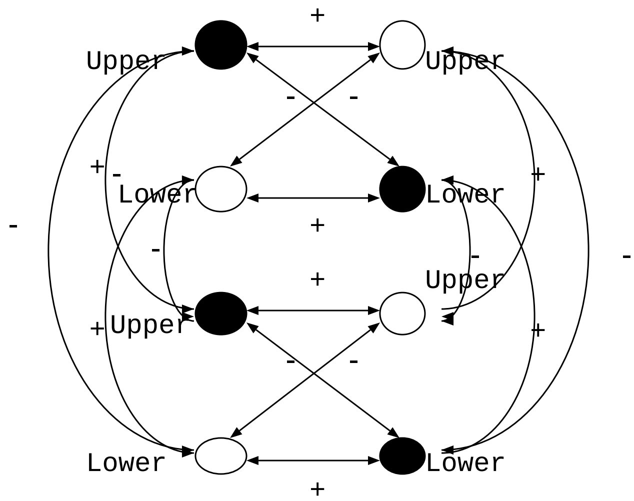

Figure 4: The State Propagation Net Transformed from Majority Net shown in Figure 2

may be needed.
(3) State-transition function may change by the local topology of units.

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