Structural Analysis on Interval Matrices by Signed Digraph: Determinant Maximization and Singularity

Yoshiteru Ishida *

Key Words - Qualitative system theory, signed directed graph, interval systems, matrix regularity

Abstract

We present some results that show the information of sign structure of matrices characterized by signed digraph can reduce computations in interval matrices. With the sign structure of the matrices, vertices of interval matrices which realizes the max/min of the determinant can be fixed. We present an algorithm to check singularity of interval matrices based on the result. We also discuss impact of this approach to other problems of interval systems.

1 Introduction

Qualitative and structural analysis, especially those by graph plays an important role when target systems are large-scale and complex. Such analysis have been studied in many different areas: economic systems, ecosystems, and system theory to mention only few. There are several different levels of abstraction in qualitative and structural analysis. One way is to focus on the level of abstraction in the parameters; how specific information is needed for describing the systems.

Systems may be divided into structural systems, sign systems and interval systems based on whether the parameters of systems are specified as zero/non-zero pattern, sign pattern and interval between two values, respectively. Researches on these different levels have been done rather independently and separately. The motivation of our research is to apply the results for one abstraction level to the other level of abstraction. In this paper, we demonstrate some results for sign systems can be effectively used to reduce the computations of interval systems.

In section 2, some concepts needed for the analysis of the determinant of interval matrix are stated. As discussed above, we use qualitative structure of the matrix expressed by signed digraph to reduce the numerical computations on interval matrices. Thus, the preliminaries mainly include results for the sign singularity of sign matrices. Section 3

*Graduate School of Information Science, Nara Institute of Science and Technology, 8916-5 Takayama, Ikoma, Nara, 630-01 Japan; email: ishida@is.aist-nara.ac.jp

presents the main result for determining the max/min values of the determinants of the interval matrices. Some vertex matrices [1] realizing max/min of the determinants of the interval matrices can be fixed based on the structural properties of signed digraph. Section 3.2 presents an algorithm to check singularity of interval matrices.

2 Sign Singularity Analysis by Signed Digraph

When the matrix is reducible, it can be reduced to irreducible components to each of which the analysis stated in this paper can be applied independently. Thus, the matrix is assumed to be irreducible in the rest of this paper. Further, for simplicity, we focus on interval matrices whose intervals can be identified as +, 0, or -. That is, we do not consider such interval as $[-2, 5]$.

Definition 2.1 (Sign Nonsingularity and Interval Nonsingularity) *Sign matrix A_s is a matrix whose elements are $+, -, 0$. Sign matrix is called sign nonsingular if all the matrices having the sign structure is nonsingular. Interval matrix A_I is a matrix whose elements are specified as interval [a,b]. Interval matrix is called interval nonsingular if all the matrix whose elements are lying within the intervals specified are nonsingular.*

An interval matrix can be considered as a set of matrices whose elements are in the intervals specified by the interval matrix. In this set of matrices, such matrix that each element take terminal value of the interval is called vertex matrix. There are 2^{n^2} vertices for the interval matrix of $n \times n$ where all the intervals $[a, b]$ of elements are specified as two distinct terminal values; $a \neq b$.

Definition 2.2 (Signed Digraph for Matrices) *Signed digraph of a matrix $A \in R^{n \times n}$ is a graph with n nodes and arc of sign +(-) is directed from node i to node j when $a_{ij} > 0 (< 0)$.*

Example 2.1

The signed directed graph of the following interval matrix is shown in Figure 1.

$$\begin{pmatrix} [-2,-1] & [-2,-1] & [0,0] \\ [-2,-1] & [-4,-3] & [-5,-4] \\ [-3,-1] & [2,4] & [-2,-1] \end{pmatrix}$$

[1] We use the word vertex for interval matrices and the word node for graphs. See the paragraph below definition 2.1.

Definition 2.3 *(Cycle and G[n]-cycle)*
The cycle of length k $c[i_1, i_2, \cdots, i_k]$ *is a path connecting the nodes* $i_1, i_2, \cdots, i_k$ *and* i_1 *sequentially*[2]. *The set of disjoint cycles is called* $G[n]$*-cycle*[3] *if the total length of these cycles is n.*

All the possible G[n]-cycles for $A \in R^{n\times n}$ correspond to all the terms in the expansion of determinant of A_I. Figure 2 shows all the possible G[3]-cycle for the graph shown in Figure 1.

Let $p[c_i]$ denote the product of all the elements in the cycle $c_i = c[i_1, i_2, \cdots, i_k]$. That is, $p[c_i] = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$. Let $C_i = \{c_{i1}, c_{i2}, \cdots, c_{iq_i}\}$ be a G[n]-cycle. Then a term in the expansion of the determinant A can be written as follows [3]:

$$(-1)^{\nu_i} p[c_{i1}] p[c_{i2}] \cdots p[c_{iq_i}]$$

where ν_i is the number of cycles with even length in C_i. Or equivalently,

$$(-1)^n (-1)^{q_i} p[c_{i1}] p[c_{i2}] \cdots p[c_{iq_i}].$$

Definition 2.4 *(Admissible Qualitative Operations)[4]*
(1) multiplying the sign in any rows by (-1).
(2) multiplying the sign in any column by (-1).
(3) interchanging any two rows.
(4) interchanging any two columns.

For any sign matrix, sign solvability (hence sign nonsingularity) is known to preserve under any combination of above admissible qualitative operations [4]. The admissible qualitative operations, however, do not preserve the structure of graph as known from the following.

[2]We use the word cycle to mean a simple cycle. That is, the nodes $i_1, i_2, \cdots, i_k$ are different.
[3]$G[n] - cycle$ has been used to explore the condition for potential stability of sign matrices [1, 2].

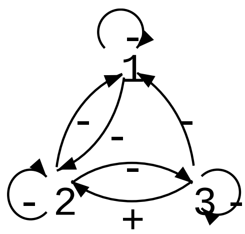

Figure 1: The Signed Digraph of the Interval Matrix

Lemma 2.1 *A cycle $c[i_1, i_2, \cdots, i_k]$ of length k can be transformed into a set of k loops (i.e. cycle with length one) $\{c[i_1], c[i_2], \cdots, c[i_k]\}$ by the admissible qualitative operations.*

[Proof]
A cycle $c[i_1, i_2, \cdots, i_k]$ can be transformed into two cycles: a loop $c[i_1]$ and a cycle $c[i_2, \cdots, i_k]$ of length k-1 by permuting i_1-th column and i_2-th column. By applying this operation to the cycle of length greater than one sequentially, the original cycle of length k will be transformed into a set of k loops. ∎

Sign structure of sign matrices has been studied extensively by Maybee[5, 6]. The following is a graphical characterization for a sign matrix to be sign nonsingular quoted from [5].

Theorem 2.2 (Maybee [5]) *Let $A \in R^{n \times n}$ be a matrix with $a_{ii} < 0$ for $i = 1, \cdots n$. Then all terms in the expansion of det A are weakly of the same sign if and only if all cycles of A are nonpositive.*

In fact, the condition that all the diagonal elements are negative can be considered to be necessary.

Lemma 2.3 *If a sign matrix is nonsingular then by the admissible qualitative operations it can be put into the form where all the diagonal elements are negative.*

[Proof]
If a sign matrix is nonsingular, its determinant must have at least one non-zero term in its expansion. (Otherwise, it will be structurally singular[4].) Then the term corresponds to a

[4]If a matrix can be determined as singular based only on its zero/non-zero pattern of its elements, then it is called structurally singular. Equivalently, structurally singular matrix can be transformed into the form which has at least one all zero column or all zero row by admissible qualitative operations.

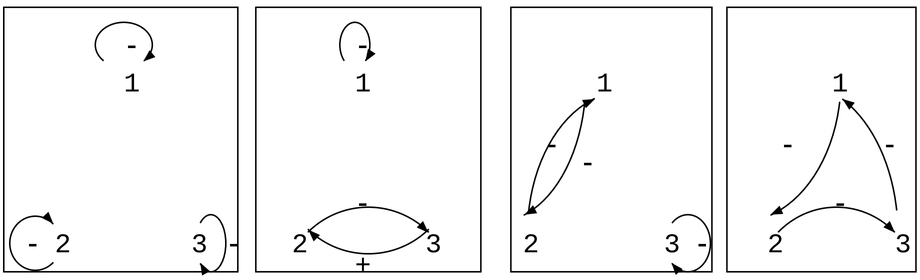

Figure 2: G[3]-Cycle Decomposition of the Signed Digraph

product of $p[i_1, i_2, \cdots, i_k]$ where for all the cycles $c[i_1, i_2, \cdots, i_k]$ in a a set of disjoint cycles whose total length is n (i.e. G[n]-cycle). By applying the admissible qualitative operation, each disjoint cycle $c[i_1, i_2, \cdots, i_k]$ of length k in the set can be transformed into a set of k disjoint loops by Lemma 2.1. ∎

Theorem 2.4 *By the admissible qualitative operations, if an interval matrix can be put into the form:*
(1) All the diagonal elements are negative, and
(2) There are no positive cycle.

In the rest of paper, matrix is assumed to be transformed into the form where all the diagonal elements are negative.

Corollary 2.5 *The sign of of a term in the expansion of the determinant of the matrix* $A \in R^{n\times n}$ *is invariant if any negative cycle of length k* $c[i_1, i_2, \cdots, i_k]$ *is replaced with corresponding k disjoint negative loops:* $\{c[i_1], c[i_2], \cdots, c[i_k]\}$ *.*

[Proof]
The sign of a term in the expansion of determinant A can be written:

$$(-1)^n(-1)^{q_i} sgn(p[c_{i1}]p[c_{i2}] \cdots p[c_{is}] \cdots p[c_{iq_i}])$$

for a G[n]-cycle $C_i = \{c_{i1}, c_{i2}, \cdots, c_{is}, \cdots, c_{iq_i}\}$. Let c_{is} represent the negative cycle of length r. Then replacing c_{is} with r disjoint negative loops $c_{is1} \cdots c_{isr}$ results in $sgn(p[c_{is}]) = - = sgn(p[c_{is1}] \cdots p[c_{isr}])$ when r is odd, $sgn(p[c_{is}]) = - = -sgn(p[c_{is1}] \cdots p[c_{isr}])$ when r is even. However, the sign of $(-1)^{q_i}$ changes when r is even while it doesn't change when r is odd. Therefore the total sign of the term does not change after the replacement. ∎

Definition 2.5 (Sign Conflict) *If all the cofactors of the element* a_{ij} *of the matrix A is not of the same sign, then the element* a_{ij} *(or its corresponding arc in the signed directed graph) is called* **sign conflict**.

The next lemma follows directly from the definition of sign conflict and G[n]-cycle.

Lemma 2.6 *If an element of the matrix is both in the G[n]-cycle consisting of only negative cycles and in the G[n]-cycle consisting of at least one positive cycle then the element is* **sign conflict**.

Whether or not the element is sign conflict can be known in the signed digraph without decomposing it into G[n]-cycle. The following is a graph theoretical condition for an element to be **sign conflict**.

Theorem 2.7 *An element a_{ij} is* **sign conflict**, *if and only if (1) the arc a_{ij} is both in a positive cycle and a negative cycle, or (2) the arc a_{ij} is in the cycle disjoint with a positive cycle.*

[Proof]
Sufficiency:
Suppose the condition (1) is satisfied. Let the length of the positive cycle and the negative cycle continuing the arc a_{ij} be k_p and k_n, respectively. Then there are two terms in the expansion of the determinant of the matrix: one consisting of the positive cycle and all the rest negative loops, the other consisting of the negative cycle and all the rest negative loops. Then the sign of the term including the positive cycle is $(-1)^{n-k_p}$ when k_p is odd and $(-1)^{n-k_p+1}$ when k_p is even as stated above, thus equal to $(-1)^{n+1}$. In the same manner, the sign of the term including the negative cycle is $(-1)^n$, thus the sign of the cofactors of a_{ij} corresponding to these terms is opposite.

When the condition (2) is satisfied, cofactors of a_{ij} corresponding to two terms: one consisting of the cycle including a_{ij}, the positive cycle disjoint with the cycle including a_{ij} and all the rest negative loops, the other consisting of the cycle including a_{ij} and all the rest negative loops, are known to have the opposite sign in the same manner as above. Hence, a_{ij} turned out to be sign conflict.
Necessity:
Suppose both conditions (1) and (2) is not satisfied. Let $C_i = \{c_{i1}, c_{i2}, \cdots, c_{iq_i}\}$ be a G[n]-cycle where the cycle including the arc ij is assumed to be arranged at c_{i1}. Then the terms including a_{ij} in the expansion of the determinant of the matrix have the following form:

$$(-1)^{\nu_i} p[c_{i1}]p[c_{i2}] \cdots p[c_{iq_i}]$$

where ν_i is the number of cycles with even length in C_i, c_{i1} is the cycle with length k including the arc ij and the other cycles $c_{i2}, \cdots c_{iq_i}$ are cycles disjoint with c_{i1}.

Since the condition (2) does not hold, all the cycles $c_{i2}, \cdots c_{iq_i}$ must be negative. By corollary 2.5 this sign is equal to the one obtained by replacing the negative cycles $c_{i2}, \cdots c_{iq_i}$ with corresponding negative loops, resulting in $(-1)^{\nu_i'} sgn(c_{i1})(-1)^{n-k}$ where ν_i' is 1 when k is even and 0 when odd. Since c_{i1} must be of the same sign for all i by the fact that (1) does not hold, all the terms including a_{ij} must be of the same sign. ∎

Although the following corollary follows directly from the definition of sign nonsingularity and sign conflict, it can also be proved by the above theorem.

Corollary 2.8 *The submatrix obtained by removing i-th row and j-th column is sign non-singular if and only if the element a_{ij} is not sign conflict.*

[Proof]
Removing i-th row and j-th column in the matrix amounts to the graph operation of

concatenating node i and node j as node [i,j] where all the incoming arc to i is then directed to the new node[i,j] and all the outgoing arcs from node j are then originated from the new node [i,j]. Thus, if this modified graph satisfies the condition of sign nonsingularity by theorem 2.2 (i.e. it does not have positive cycle) then the original graph must not satisfy neither (1) nor (2) of theorem 2.7.

Suppose a_{ij} is not sign conflict, then the graph must not satisfy neither (1) nor (2). That is all the cycles containing a_{ij} are of the same sign and all the other cycles are negative sign. Since the sign of all the cycles containing a_{ij} can be changed without affecting the sign of all the other cycles disjoint with the cycles containing a_{ij} by multiplying the sign of i-th row (or j-th column) by -1, concatenating node i and node j result in the graph of sign nonsingular after admissible qualitative operations. ∎

In other words, sign nonsingular matrix is such a matrix that does not have any sign conflict element. The next corollary follows from the theorem 2.7.

The element a_{12} of the matrix of Example 2.1 is sign conflict, since it satisfies the condition (1) of theorem 2.7 (it is included in the positive cycle $c[1,2]$ and the negative cycle $c[1,2,3]$). The element a_{33} is also known to be sign conflict, since it satisfies the condition (2) of theorem 2.7 (there is a positive cycle e $c[1,2]$ disjoint with it). These are also known to be sign conflict by applying above Lemma 2.6 to the G[3]-cycle decomposition shown in Figure 2.

3 Analysis on Interval Matrices

3.1 Min/Max of the det A of interval matrices

Since sign nonsingularity requires that all the non-zero terms of the expansion of determinant must be of the same sign, the next lemma follows immediately.

Lemma 3.1 *If the sign digraph of an interval matrix is sign nonsingular, then the vertex that realizes the minimum absolute value of determinant of the interval matrix is that with smaller(greater) absolute value of two terminal values for each interval.*

Since all the diagonal elements of the interval matrices under consideration are assumed to be negative, the determinant of the interval matrices have the term $p[c_1]p[c_2]\cdots p[c_n]$ in the expansion. We call the sign of the term $sgn(p[c_1]p[c_2]\cdots p[c_n]) = (-1)^n$ standard sign, since all the other non-zero terms in the determinant expansion of sign nonsingular matrices have the same sign as this.

Even if an interval matrix is not sign nonsingular, the terminal value that realizes the maximum or minimum absolute value can be determined if the element is sign non-conflict.

Theorem 3.2 *The vertex that realizes the minimum absolute value of determinant of the interval matrix is that with smaller(greater) absolute value for the sign non-conflict element when it is in the negative(positive) cycle.*

[Proof]
If the arc corresponding to the sign non-conflict element is in the negative cycle then the term of determinant expansion including the cycle has the same as the standard sign. Thus, the minimum absolute value of determinant is realized at the terminal of greater absolute value in the same manner as the case of sign nonsingular stated in lemma 3.1. When the sign non-conflict element is in the positive cycle, corresponding term of determinant expansion is opposite to the standard sign, hence it must take smaller absolute value for the maximum absolute value of the determinant. ∎

Obviously, results similar to lemma 3.1 and theorem 3.2 stating the vertex realizing the maximum absolute value can be obtained with the word "smaller" and "greater" exchanged.

3.2 Algorithm for finding minimum value of the determinant

Based on above theorem 3.2, the following algorithm for finding the vertex that realizes determinant with minimum absolute value is proposed.

Algorithm 3.3

STEP 1: Assign the terminal values to the elements of sign non-conflict based on the theorem 3.2.

STEP 2: Find the element of sign conflict whose cofactor does not have the element of sign conflict. If found, assign the appropriate terminal value to the element of sign conflict depending on the sign of the cofactor. Continue this step until there is no element of sign conflict whose cofactor does not have the element of sign conflict.

STEP 3: Find the element of sign conflict whose cofactor has the elements of sign conflict, but the sign of the cofactor does not change which terminal value the element of sign conflict may take. If found, assign the appropriate terminal value to the element of sign conflict depending on the sign of the cofactor, and go back to the STEP 2. If not found, proceed to the next step.

STEP 4: Carry out a combinatorial search for the remained elements of sign conflict.

Example 3.1

In the following, the above algorithm is demonstrated for the same example as 2.1

In STEP 1 of the algorithm, a_{12} is the sign conflict element by the condition (1) of theorem 2.7. a_{33} is also the sign conflict element by the condition (2) of the theorem. All the other elements are sign non-conflict and their terminal value can be determined. Sign conflict element is indicated by * symbol in the matrix.

Since a_{33} has the sign conflict element a_{12} in its disjoint cycle, and a_{12} has the sign conflict element a_{33} in its disjoint cycle, there is no sign conflict element specified in the STEP 2. The terminal values so far determined for the sign non-conflict elements are underlined in the matrix.

In STEP 3, the cofactor of a_{12}, i.e., $(-4) \times (-1) - (-2) \times [-2, -1]$ does not change the sign whichever the terminal value $a_{33} = [-2, -1]$ may take. Hence the terminal value of a_{12} can be determined. Since there is no more sign conflict other than a_{33}, the terminal value of a_{33} can be assigned in the STEP 2. Thus, the terminal value of -1 at a_{12} is taken for the minimum absolute value of the determinant. Then, this will again determine the terminal value of -1 at a_{33}. Thus the vertex that realize minimum absolute value of determinant is obtained. The value the determinant is -13, hence the interval matrix is nonsingular.

$$\begin{pmatrix} [-2, \underline{-1}] & [-2, -1]^* & [0, 0] \\ [\underline{-2}, -1] & [-4, \underline{-3}] & [-5, \underline{-4}] \\ [-3, \underline{-1}] & [\underline{2}, 4] & [-2, -1]^* \end{pmatrix}$$

4 Discussions

Checking singularity of interval matrices is known to be NP-hard [7]. One practical way of reducing computations in such problems is to use qualitative and structural characters embedded, which may be exposed by graphs.

Comparing with the Rohn's algorithm [8], our algorithm considers the sign nonsingular system as a basis and other elements as disturbance elements. On the other hand, Rohn's algorithm considers the center matrix as a basis and radius matrix as the disturbing elements.

The graphical analysis stated in this paper can be applied before carrying out any algorithms for checking singularity of the interval matrices such as [8], which would reduce the calculation done in the algorithms.

The signed digraph analysis can be used not only to check the singularity of given interval matrix by only sign pattern, but also it will direct which submatrices to calculate rather than calculating 2^{2n-1} matrices of full size[5] [8, 9].

[5] Rohn has shown that checking singularity of 2^{2n-1} vertices will suffice for interval singularity rather than checking all the vertices 2^{n^2}.

Although our method requires finding cycles in the graph which is also computationally expensive, computer simulations (the results of computer simulation are omitted in this paper due to the page limitation) show that our method is more efficient than by those so far proposed [8, 9], for large-scale matrices especially those with many zero elements and hence small number of cycles. It should be stressed that result by the graphical analysis in our algorithm (i.e. whether or not a given interval matrix is sign nonsingular, and whether or not each element of the matrix is sign conflict,) can be used not only for the given particular interval matrix, but for all interval matrices that have same sign structure as the given one. Further, the results by graphical analysis can be used for sensitivity analysis; it can tell which terminal values can be changed keeping nonsingularity, and which terminal values contribute to the maximum or minimum value of the determinant.

It is expected that our approach of using qualitative and structural information to reduce combinations of interval matrices may be applied to many other similar problems: solvability of static interval systems, stability, initial value problem of dynamic interval systems, and so on. These problems are currently under investigation.

5 Conclusion

Computations about interval matrices often require an computationally expensive combinatorial search. Finding the vertices of interval matrices that realizes max/min determinant and checking singularity of interval matrices are two examples of such computation. We proposed a method incorporating the features of sign structure obtained by the analysis on signed digraph of matrices. We have shown that some elements of the matrix, originally specified to lie in an interval, can be fixed at the terminal value for finding max/min determinant of the interval matrices. Further, the analysis on signed digraph not only reduces the computation but also suggests the structural properties such that which terminal values of the interval contribute for max/min of the determinant and which elements are sensitive for interval singularity.

Implementation of the algorithm and precise numerical evaluation of how much our method reduce the computation of interval matrices will be discussed elsewhere. Similar approach to the other computations about interval matrices such as stability analysis and solving interval systems are currently under investigation.

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